

A note on the relation between joint and differential invariants

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Abstract

We discuss the general properties of the theory of joint invariants of a smooth Lie group action in a manifold. Many of the known results about differential invariants, including Lie's finiteness theorem, have simpler versions in the context of joint invariants. We explore the relation between joint and differential invariants, and we expose a general method that allow to compute differential invariants from joint invariants.

1 Introduction

Let us consider a connected Lie group G and an smooth G -manifold M ,

$$\alpha: G \times M \rightarrow M, \quad (g, p) \mapsto \alpha(g, p) = gp.$$

For each $g \in G$ we denote by α_g the diffeomorphism that maps each point p of M to gp . The action of G extends diagonally to all the Cartesian powers M^k of M by setting for each k -tuple $\bar{p} = (p_1, \dots, p_k)$,

$$g\bar{p} = (gp_1, \dots, gp_k).$$

A smooth *k*-joint invariant (or joint invariant of k points) of the action of G in M defined in $W \subseteq M^k$ is a function $I \in \mathcal{C}_{M^k}^\infty(W)$ such that for each k -tuple $\bar{p} \in W$ and each $g \in G$ such that $g\bar{p} \in W$ we have $I(\bar{p}) = I(g\bar{p})$.

Joint invariants appear frequently in classical geometry (see [9, 10]). Some well know examples are: affine ratio in affine geometry; anharmonic ratio in projective geometry; distance of two points, area of a triangle, volume of a tetrahedron in Euclidean geometry. There is also a natural relation between joint invariants and the geometry of differential equations. The joint invariants of the Lie-Vessiot-Guldberg algebra of an ordinary differential equation give rise to non-linear superposition formulas. Such relation has been studied recently by several authors, [1, 2, 3]. The most know example is the conservation of the anharmonic ratio by the Riccati equation that gives rise to the non-linear superposition of three different solutions.

On the other hand, the action of the group G in M , also prolongs to the Weil near-point¹ bundles $T^{m,r}M$, of Taylor developments of order r of maps $(\mathbb{R}^m, 0) \rightarrow M$. Let $s: (\mathbb{R}^m, 0) \rightarrow M$ is a germ smooth map, and $j_0^r(s) \in T^{m,r}M$ his Taylor development of order r at 0. For each $g \in G$ with we can define,

$$g(j_0^r(s)) = j_0^r(\alpha_g \circ s).$$

A smooth *differential invariant* of order r and rank m of the action of G in M is a function I defined in an open subset $W \subseteq T^{m,r}M$ that is invariant by the induced action of G in $T^{m,r}M$. Differential invariants play an interesting role in the analysis of differential equations, the non-linear differential Galois theory, and G -structures.

There are some known natural relations between joint and differential invariants. For instance, let us consider a diffeomorphism f of the projective line and $\{\sigma_t\}_{t \in \mathbb{R}}$ a monoparametric group of transformations. It is well know (see for instance [11], page 10) that the Schwartzian derivative can be seen as the infinitesimal deformation of the anharmonic ratio in the following way. Let x be any point, then

$$\begin{aligned} & \frac{(f(x) - f(\sigma_{2\varepsilon}x))(f(\sigma_\varepsilon)x - f(\sigma_{3\varepsilon}x))}{(f(x) - f(\sigma_\varepsilon x))(f(\sigma_{2\varepsilon}x) - f(\sigma_{3\varepsilon}x))} = \\ & \frac{(x - \sigma_{2\varepsilon}x)(\sigma_\varepsilon x - \sigma_{3\varepsilon}x)}{(x - \sigma_\varepsilon x)(\sigma_{2\varepsilon}x - \sigma_{3\varepsilon}x)} + \frac{3f''(x)^2 - 2f'(x)f'''(x)}{f'(x)^2} \varepsilon^2 + o(\varepsilon^3) \end{aligned}$$

In this article we study the general theory of joint invariants, and explore the relation between joint and differential invariants. We show that there is a general mechanism of derivation of joint invariants that yields differential invariants.

2 Joint invariants

2.1 Sheaf of local invariants

For each $p \in M$ we denote by Gp the orbit of p ,

$$Gp = \{gp \in M \mid g \in G\}.$$

Let $U \subseteq M$ be an open subset. An smooth function $I \in \mathcal{C}_M^\infty(U)$ is called an invariant if for each $p \in U$ and each $g \in G$ such that $gp \in U$ we have $I(p) = I(gp)$. In other words, for each $p \in U$ we have that $I|_{Gp \cap U}$ is a constant function. The functor that assigns to each open subset U the set of

¹Some authors prefer to define differential invariants in the Jet bundles, that are algebraic quotients of the near-point bundles. We prefer here the formalism of Weil near-point bundles because it adapt better to our computations. A comparison between those notions, and detailed explanations about the construction of Jet and Weil bundles can be found in [7].

the invariants defined in U is a presheaf. Its associated sheaf \mathcal{A}^G is called the *sheaf of local invariants* of the action of G in M . We have thus,

$$\mathcal{A}^G(U) = \{I \in \mathcal{C}_M^\infty(U) \mid \forall p \in U \text{ } I \text{ is locally constant in } Gp \cap U\}.$$

If U is union of orbits of the action of the connected Lie group G then $\mathcal{A}^G(U)$ is exactly the ring of invariants of the action of G in U . For a general open subset U the ring of local invariants $\mathcal{A}^G(U)$ may be bigger than the ring of invariants.

Example 1 Let us consider the group $\text{PGL}(2, \mathbb{R})$ of classes modulo scalars of non-degenerated 2×2 matrices. The Möbius action of $\text{PGL}(2, \mathbb{R})$ in \mathbb{RP}_1 is given in the canonical affine coordinate x in \mathbb{RP}_1 by the formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} x = \frac{ax + b}{cx + d}.$$

Let us consider the subgroups $\text{Mov}(1, \mathbb{R}) \subset \text{Aff}(1, \mathbb{R}) \subset \text{PGL}(2, \mathbb{R})$ where the respective inclusions are explicitly given as:

$$\left\{ \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} : \lambda \in \mathbb{R} \right\} \subset \left\{ \begin{bmatrix} \lambda & \mu \\ 0 & 1 \end{bmatrix} : \lambda, \mu \in \mathbb{R}, \lambda \neq 0 \right\} \subset \text{PGL}(2, \mathbb{R}).$$

We have:

- (a) The oriented distance $d^+(x_1, x_2) = x_2 - x_1$ is a 2-joint invariant in the open subset $\mathbb{R}^2 \subset (\mathbb{RP}_1)^2$ of the action of $\text{Mov}(1, \mathbb{R})$ in \mathbb{RP}_1 .
- (b) The affine ratio $[x_1, x_2; x_3] = \frac{x_3 - x_1}{x_2 - x_1}$ is a 3-joint invariant in the open subset $\mathbb{R}^3 \setminus \{x_2 = x_1\} \subset (\mathbb{RP}_1)^3$ of the action of $\text{Aff}(1)$ in \mathbb{RP}_1 .
- (c) The anharmonic ratio $[x_1, x_2, x_3; x_4] = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)}$ is a 4-joint invariant in the open subset $(\mathbb{RP}_1)^4 \setminus \{x_i = x_j\}_{i \neq j} \subset (\mathbb{RP}_1)^4$ of the action of $\text{PGL}(2)$ in \mathbb{RP}_1 .

Example 2 Let us consider the group $\text{Mov}(n, \mathbb{R})$ of Euclidean motions in \mathbb{R}^n . Let us write $\bar{p} = (p_1, \dots, p_k)$ and $p_i = (x_{1,i}, \dots, x_{n,i})$.

- (a) The square distance $d^2(p_1, p_2) = \sum_{i=1}^n (x_{i,1} - x_{i,2})^2$ is a smooth 2-joint invariant defined in $(\mathbb{R}^n)^2$.
- (b) The Euclidean area $A(p_1, p_2, p_3)$ of a triangle of vertices p_1, p_2, p_3 is a smooth 3-joint invariant defined in the open subset of 3-tuples of points in general position. Let us note that this set is dense if $n \geq 2$ and empty for $n < 2$.
- (c) The Euclidean volume $V(p_1, p_2, p_3, p_4)$ of tetrahedron of vertices p_1, p_2, p_3, p_4 is a smooth 4-joint invariant defined in the open subset of 4-tuples of points in general position. Let us note that this set is dense if $n \geq 3$ and empty for $n < 3$.

- (d) The k -dimensional Euclidean volume $V_k(p_1, \dots, p_{k+1})$ of a k -simplex of vertices p_1, \dots, p_{k+1} is a smooth $(k+1)$ -joint invariant defined in the open subset of $(k+1)$ -tuples of points in general position. Let us note this set is dense if $n > k$ and empty for $n \leq k$.

Infinitesimal generators

Let us denote by \mathcal{G} the Lie algebra of right invariant vector fields in G . There is a natural Lie algebra morphism from \mathcal{G} into the Lie algebra $\mathfrak{X}(M)$ of vector fields in M :

$$\mathbf{ig}: \mathcal{G} \rightarrow \mathfrak{X}(M), \quad X \mapsto \mathbf{ig}(X), \quad \mathbf{ig}(X)_p = \left. \frac{d(e^{tX}p)}{dt} \right|_{t=0}.$$

We call infinitesimal generators of the action of G in M to the vector fields in the image of \mathbf{ig} . The sheaf of local invariants \mathcal{A}^G is easily characterized as the sheaf of first integral of the Lie algebra of the infinitesimal generators in M .

$$\mathcal{A}^G(U) = \{f \in \mathcal{C}^\infty(U), \mid \forall X \in \mathcal{G} \ \mathbf{ig}(X)f = 0\}.$$

The following result tell us that we can restrict our consideration to dense open subsets of M .

Lemma 1 *Let $U \subseteq W \subseteq M$ be open subsets in M such that U is dense in W . Then if I is a function in W such that $I|_U$ is a local invariant, then I is local invariant in W .*

Proof. Let A_1, \dots, A_r be a basis of \mathcal{G} . Then, I is a local invariant in U if and only if $\mathbf{ig}(A_i)I = 0$ in U for $i = 1, \dots, r$. By continuity we have that $\mathbf{ig}(A_i)I = 0$ in W and thus I is a local invariant in W . \square

It is also useful to consider the distribution of vector fields spanned by the infinitesimal generators. We denote by \mathcal{L}^G the distribution of vector fields that assigns to each point p of M the vector space \mathcal{L}_p^G ,

$$\mathcal{L}_p^G = \text{span}\{\mathbf{ig}(X)_p \mid X \in \mathcal{G}\} \subseteq T_p M,$$

it is clear that $T_p(Gp) = \mathcal{L}_p^G$. The distribution \mathcal{L}^G is, by definition, stable by the Lie bracket. It determines a unique Pfaff system Σ^G such that $(\Sigma^G)^\perp = \mathcal{L}^G$. We review the basic definitions about Pfaff systems and their first integrals in the next section.

2.2 Some considerations about first integrals

We denote by Ω_M^\bullet the sheaf of differential exterior forms in M . Thus,

$$\Omega_M^\bullet(U) = \bigoplus_{k=0}^{\dim(M)} \Omega_M^k(U).$$

For open $p \subset M$ the stalk $\Omega_{M,p}^\bullet$ is endowed with the wedge product and the exterior differential it is a free $\mathcal{C}_{M,p}^\infty$ module of rank $2^{\dim(M)}$. Thus, Ω_M^\bullet is a sheaf of differential rings, and also a locally free of finite rank sheaf of \mathcal{C}_M^∞ -modules. Given a set $S \subset \Omega_{M,p}^\bullet$ we will denote by (S) the ideal of $\Omega_{M,p}^\bullet$ spanned by S and by $\{S\} = (S, dS)$ the differential ideal of $\Omega_{M,p}^\bullet$ spanned by S . The same notation extends easily for a subsheaf $\mathcal{S} \subset \Omega_M^\bullet$.

Definition 1 An exterior differential system (e.d.s.) Σ in M is a sheaf $\Sigma \subset \Omega_M^\bullet$ of differential ideals without zero-order equations. That is, for each $p \in M$ it satisfies $\Sigma_p \cap \Omega_{M,p}^0 = \{0\}$.

Definition 2 A differential exterior system Σ in M is a Pfaff system if it is spanned, as a sheaf of differential ideals, by its sections of degree one, that is for each $p \in M$, $\Sigma_p = \{\Sigma_p \cap \Omega_{M,p}^1\}$.

Definition 3 Let U be an open subset in M . A function $F \in \mathcal{C}^\infty(U)$ is called a first integral of Σ if $dF \in \Sigma(U)$. The first integrals of Σ form a sheaf of smooth functions that we denote by $\mathbf{int}(\Sigma)$.

First integrals commute with smooth maps. If $\varphi: N \rightarrow M$ is smooth and Σ is an e.d.s. in M , then $\mathbf{int}(\varphi^*\Sigma) = \varphi^*(\mathbf{int}(\Sigma))$. Given a sheaf $\mathcal{S} \subset \mathcal{C}_M^\infty$ of rings of smooth functions in M we can differentiate it to obtain a Pfaff system $d\mathcal{S}$,

$$(d\mathcal{S})_p = \{d(\mathcal{S}_p)\} \subset \Omega_{M,p}^\bullet.$$

It is clear that all sections of \mathcal{S} are first integrals of $d\mathcal{S}$,

$$\mathcal{S} \subseteq \mathbf{int}(d\mathcal{S}).$$

And, for a Pfaff system Σ , the differential of first integrals of Σ are by definition in Σ ,

$$d(\mathbf{int}(\Sigma)) \subseteq \Sigma.$$

Definition 4 A Pfaff system Σ is called *integrable* if it is completely determined by its sheaf of first integrals, $d(\mathbf{int}(\Sigma)) = \Sigma$.

A Pfaff system Σ has a associated several distribution of vector fields Σ^\perp and Σ' called the orthogonal and de characteristic distribution. For each $p \in M$ we have:

$$\begin{aligned} \Sigma_p^\perp &= \{X_p \in T_p M \mid \forall \omega \in \Sigma_p \cap \Omega_p^1 \quad \omega_p(X_p) = 0\} \\ \Sigma'_p &= \{X_p \in T_p M \mid \forall \omega \in \Sigma_p \quad \mathbf{i}_{X_p} \omega_p = 0\} \end{aligned}$$

It is clear that $\Sigma' \subseteq \Sigma^\perp$. The distribution Σ' stable by Lie brackets.

Definition 5 A Pfaff system Σ is called regular of rank r if for each $p \in M$ there are $\omega_1, \dots, \omega_r \in \Sigma_p$ such that:

$$(a) \quad \Sigma_p = \{\omega_1, \dots, \omega_r\}.$$

(b) $\omega_1(p), \dots, \omega_r(p) \in T_p^*M$ are linearly independent.

It is clear that Σ is a regular Pfaff system of rank r if and only if the distribution of vector fields Σ^\perp is a regular sub-bundle of TM of rank $\dim(M) - r$. Thus, regular distributions of vector fields and regular Pfaff systems in M are in one-to-one correspondence: a 1-form ω is a section of Σ if and only if it vanishes along Σ^\perp . It is also interesting to remark that for any regular submersion $\varphi: N \rightarrow M$ we have $\varphi^*(\Sigma)$ is a regular and:

$$\varphi^*(\Sigma)^\perp = d\varphi^{-1}(\Sigma^\perp).$$

The following result is well known:

Theorem 1 (Frobenius) *Let Σ be a regular Pfaff system in M of rank r . The following are equivalent:*

- (a) Σ is integrable.
- (b) Σ^\perp is stable by Lie bracket.
- (c) $\Sigma^\perp = \Sigma'$.
- (d) Σ is spanned as a sheaf of ideals its homogeneous component of degree one:
 $\Sigma = (\Sigma \cap \Omega_M^1)$.
- (e) Through each point $p \in M$ it passes a submanifold S of dimension $\dim M - r$ such that for each $q \in S$, $T_q S = \Sigma_q^\perp$.
- (f) For each point $p \in M$ there are r functionally independent first integrals of Σ in $\mathcal{C}_{M,p}^\infty$.

A system of r functionally independent first integrals of Σ , defined on an open subset U , is called a *complete system* of first integrals of Σ in U . Any other first integral of Σ is locally functionally dependent of them.

2.3 Functional dependence

Let us consider \mathcal{R} and \mathcal{S} two sheaves of smooth functions on M . We can consider the sheaf of functions that locally depend functionally of sections of \mathcal{R} and \mathcal{S} . The theorem of functional dependence says that a germ of smooth function $f \in \mathcal{C}_{M,p}^\infty$ is function of germs $f_1, \dots, f_k \in \mathcal{R}_p$, $h_1, \dots, h_s \in \mathcal{S}_p$ if there are $\lambda_1, \dots, \lambda_{k+s} \in \mathcal{C}_{M,p}^\infty$ such that

$$df = \sum_{i=1}^k \lambda_i df_i + \sum_{j=1}^s \lambda_{k+j} dh_j.$$

Thus, a function $f \in \mathcal{C}^\infty(U)$ is locally functionally dependent of those in $\mathcal{R}(U)$ and $\mathcal{S}(U)$ if and only if f is a simultaneously first integral of the Pfaff systems $d\mathcal{R}$ and $d\mathcal{S}$. This consideration allow us to give a simpler definition:

Definition 6 The sheaf $\mathcal{R} \odot \mathcal{S}$ of functions that locally depend functionally of those in \mathcal{R} and \mathcal{S} is:

$$\mathcal{R} \odot \mathcal{S} = \mathbf{int}(\{d\mathcal{S}, d\mathcal{R}\})$$

The following result list some direct consequences of the definitions and does not need a proof.

Lemma 2 *Let Σ_1 and Σ_2 be two regular Pfaff systems in M such that $\{\Sigma_1, \Sigma_2\}$ is also regular, and let $\mathcal{R}_1, \mathcal{R}_2$ be their respective sheaves of first integrals.*

- (a) *$\{\Sigma_1, \Sigma_2\}$ is integrable and $\mathcal{R}_1 \odot \mathcal{R}_2$ is its sheaf of first integrals.*
- (b) *$f \in \mathcal{C}_M^\infty(U)$ for U open subset of M is a first integral of $\{\Sigma_1, \Sigma_2\}$ if and only if df vanish on the distribution of vector fields $\Sigma_1^\perp \cap \Sigma_2^\perp$.*

2.4 Lifting of joint invariants

From now on let us fix some notation about the sheaves of local joint invariants. Let us denote $\mathcal{A}^{G,k} \subset \mathcal{C}_{M^k}^\infty$ the sheaf of local k -joint invariants, $\mathcal{L}^{G,k}$ the distribution of vector fields spanned by the infinitesimal generators of the action of G in M^k and $\Sigma^{G,k}$ the Pfaff system generated by 1-forms in open sets of M^k vanishing on $\mathcal{L}^{G,k}$.

By a *generating system of (local) k -joint invariants* we mean a system of first integrals of $\Sigma^{G,k}$. If we work in an open subset of M^k in which $\mathcal{L}^{G,k}$ is a regular distribution of vector fields of the same dimension than G , the Frobenius theorems ensures that all generating systems of (local) k -joint invariants consist of $k \dim(M) - \dim(G)$ functionally independent invariants.

Lemma 3 *Assume that M^k contains an open subset W_k in which $\Sigma^{G,k}$ is regular. Let $U \subseteq M^k$ be an open subset. Any smooth function $I \in \mathcal{C}^\infty(U)$ with $U \subset M^k$ is a local k -joint invariant if and only if $I|_{W_k \cap U}$ is a first integral of $\Sigma^{G,k}$.*

Proof. If $\Sigma^{G,k}$ is regular in W_k , a first integral in W_k is a function which is locally constant along the orbits, and thus, a local k -joint invariant. Then, by Lemma 1, we conclude. \square

We also denote by π_i^k the projection from M^k to M^{k-1} that consists in dropping the i -th component:

$$\pi_i^k: M^k \rightarrow M^{k-1}, \quad (p_1, \dots, p_k) \mapsto (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_k).$$

These projections π_i^k are G -equivariant, and thus:

$$\pi_i^{k*}(\mathcal{A}^{G,k-1}) \subseteq \mathcal{A}^{G,k}.$$

This inclusion simply means that each $(k-1)$ -joint invariant can be seen as a k -joint invariant in k different ways by dropping one of the arguments.

Let $W \subset M^k$ be an open subset. Let us consider a function $I \in \mathcal{C}_{M^k}^\infty(W)$ that depends functionally on local joint invariants of $k - 1$ points:

$$I \in (\pi_1^{k*} \mathcal{A}^{G,k-1} \odot \dots \odot \pi_k^{k*} \mathcal{A}^{G,k-1})(W),$$

then it is clear that it is a local joint invariant of k points, thus:

$$\pi_1^{k*} \mathcal{A}^{G,k-1} \odot \dots \odot \pi_k^{k*} \mathcal{A}^{G,k-1} \subseteq \mathcal{A}^{G,k}.$$

We can ask if any local joint invariant of k points can be expressed as a function of local joint invariants of less than k points.

Example 3 Let us consider the action of $\text{PGL}(2, \mathbb{R})$ and its subgroups in \mathbb{RP}_1 discussed in Example 1. Since \mathbb{RP}_1 is 1-dimensional, and by two different liftings $\pi_i^k: \mathbb{RP}_1^k \rightarrow \mathbb{RP}_1^{k-1}$ we will always give functionally independent joint invariants, we can conclude:

- (a) Any k -joint invariant of $\text{Mov}(1, \mathbb{R})$ with $k \geq 2$ is functionally dependent of the liftings of the oriented distance.
- (b) Any k -joint invariant of $\text{Aff}(1, \mathbb{R})$ with $k \geq 3$ is functionally dependent of the liftings of the affine ratio
- (a) Any k -joint invariant of $\text{PGL}(2, \mathbb{R})$ with $k \geq 4$ is functionally dependent of the liftings of the anharmonic ratio.

Example 4 Let us discuss the functional dependence of the joint invariants shown in Example 2. Let $\bar{p} = (p_1, \dots, p_k)$ be a tuple in $(\mathbb{R}^n)^k$. By iterating $k - 2$ times the liftings from $(\mathbb{R}^n)^2$ up to $(\mathbb{R}^n)^k$ we obtain $\binom{k}{2}$ functions called *mutual distances*:

$$d_{ij}(\bar{p}) = d(p_i, p_j).$$

For $k \leq n$ the mutual distances are functionally independent in a dense open subset of $(\mathbb{R}^n)^k$. For $k = n$ we have that the rank of $(\Sigma^{G,n})^\perp$ coincides with the dimension of $\text{Mov}(\mathbb{R}, n)$ which is $\binom{n+1}{2}$. In this case the mutual distances form a complete system of $\binom{n}{2}$ first integrals, the dimension of the manifold of configurations of n points is $(\mathbb{R}^n)^n$ is $n^2 = \binom{n}{2} + \binom{n+1}{2}$, the sum of the number of independent n -joint invariants and the dimension of the group. The functional dependence relations between the mutual distances and the rest of the k -joint invariants discussed in Example 2 are well known (see [4] sec. 3.6.1) theorems of Euclidean geometry, namely:

- (a) The area of a triangle $A(p_1, p_2, p_3)$ is a function of d_{12} , d_{23} , and d_{31} by means of the Heron formula:

$$A = \frac{1}{4} \sqrt{(d_{12} + d_{13} + d_{23})(-d_{12} + d_{13} + d_{23})(d_{12} - d_{13} + d_{23})(d_{12} + d_{13} - d_{23})}.$$

We shall also remark that the classical theorems of trigonometry, the sinus and cosinus theorems, are also relations of functional dependence between 3-joint invariants of the Euclidean plane geometry.

- (b) The volumen of the tetrahedron $V(p_1, p_2, p_3, p_4)$ is a function of the functions d_{ij} for $i, j = 1, \dots, 4$ by means of thr Cayley-Menger determinant:

$$V = \frac{1}{12\sqrt{2}} \sqrt{\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{13}^2 & d_{23}^2 & 0 & d_{34}^2 \\ 1 & d_{14}^2 & d_{24}^2 & d_{34}^2 & 0 \end{vmatrix}}$$

- (c) The k -dimensional volume satisfies the equation:

$$2^k (k!)^2 V_k^2 = (-1)^{k+1} \det(A),$$

where C is the $(k+1) \times (k+1)$ matrix whose elements c_{ij} are:

$$c_{ij} = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j = 1 \text{ or } j \neq i = 1, \\ d_{i-1, j-1}^2 & \text{in any other case.} \end{cases}$$

2.5 Finiteness theorem

For each p in M we denote by $\text{Est}(p)$ the stabilizer group of p ,

$$\text{Est}(p) = \{g \in G \mid gp = p\}$$

and by $\text{est}(p)$ its Lie algebra. With respect to the action of G on k -tuples, it is useful to note that for $\bar{p} = (p_1, \dots, p_k)$,

$$\text{Est}(\bar{p}) = \bigcap_{i=1}^k \text{Est}(p_i), \quad \text{est}(\bar{p}) = \bigcap_{i=1}^k \text{est}(p_i).$$

The action of G in M is called *locally faithful* if for each open subet $U \in M$ the map:

$$\mathbf{ig}|_U: \mathcal{G} \rightarrow \mathfrak{X}(U), \quad A \mapsto \mathbf{ig}(A)|_U$$

is injective. It is equivalent to say that for each $A \in \mathcal{G}$ the set of zeroes of $\mathbf{ig}(A)$ has empty interior. It is clear that an algebraic or analytic faithful action is also locally faithful, but an smooth faithful action can be not locally faithful.

A point $p \in M$ is said *locally regular* if $\text{est}(p) = \{0\}$. The set of locally regular points is an open subset of M . We will say that the action of G in M is *generically locally regular* if the set of locally regular points is dense in M .

It is clear that if the action of G in M^k is generically locally regular, then the action of G in M^{k+1} is also generically locally regular. If there is an smaller natural number k_0 such that the action of G in M^{k_0} is generically locally regular, we say that k_0 is the *local rank* of the action of G in M . For $k > k_0$ the Pfaff system $\Sigma^{k,G}$, restricted to the dense open subset of locally regular points, is a regular integrable system.

Lemma 4 *If the action of G in M is locally faithful then the action of G in M has a finite local rank k_0 . It verifies:*

$$\frac{\dim G}{\dim M} \geq k_0 \geq \dim G.$$

Proof. Let $\bar{p} = (p_1, \dots, p_k) \in M^k$. The vector space $\text{est}(\bar{p})$ is by definition the kernel of a linear map from \mathcal{G} to $T_{\bar{p}}(M^k)$. The k -tuple \bar{p} is regular if and only if this morphism is injective. This is only possible if the dimension of the target space is bigger than the dimension of the source space. This gives us the left side of the inequality of the statement.

Let us see that the set of locally regular k -tuples, with $k \geq \dim G$ is dense. Let us consider $\bar{p} = (p_1, \dots, p_k) \in M^k$, and U any neighbourhood of \bar{p} , without loss of generality we can assume that $U = U_1, \dots, U_k$ where each U_i is a neighbourhood of p_i . Since the action is locally faithful there is p_1^* in U_1 such that the inclusion $\text{est}(p_1^*) \subset \mathcal{G}$ is strict. Let us denote this space as E_1 . Then for each $i = 1, \dots, k-1$ we set the space E_k in the following manner:

- (a) If $E_i = \{0\}$ we set $p_{i+1}^* = p_{i+1}$ and $E_{i+1} = \{0\}$.
- (b) If $E_i \neq \{0\}$ we take a non-zero vector $A_i \in E_i$. By hypothesis the set of zeroes of $\mathbf{ig}(A_i)$ is of empty interior in U_{i+1} then, we set p_{i+1}^* a point in U_2 such that $\mathbf{ig}(A_i)(p_{i+1}^*) \neq 0$ and $E_{i+1} = \text{est}(p_1^*, \dots, p_{i+1}^*)$. In this case we have $A_i \notin E_{i+1}$, and thus an strict inclusion $E_i \supset E_{i+1}$.

By the above argument, we have defined a descending chain of vector spaces:

$$\mathcal{G} \supseteq E_1 \supseteq \dots \supseteq E_k,$$

where each inclusion is strict until the chain stabilizes at 0. It follows that for $k \geq \dim G$ then $E_k = 0$, thus:

$$\text{est}(p_1^*, \dots, p_k^*) = E_k = 0,$$

and $\bar{p}^* = (p_1^*, \dots, p_k^*)$ is a regular k -tuple un U . We have proven that for $k \geq \dim G$ the action of G in M^k is generically regular. \square

Remark 1 The left inequality of the statement is treated by S. Lie in the study of superposition laws, in such context it is know as Lie inequality. This local rank coincides is the number of known solutions necessary to describe the general solution in a o.d.e. admitting non linear superposition laws. The reader may consult [1, 2, 3] for further explanations.

Proposition 1 *Assume that $\Sigma^{G, k-1}$ is a regular integrable Pfaff system in some dense open subset W_{k-1} of M^{k+1} and that there is an dense open subset W_k in M^k such that for each k -tuple $\bar{p} \in W_k$ we have:*

$$\bigcap_{i=1}^k \left(\mathcal{L}_{\bar{p}}^{G, k} \oplus \ker(d_{\bar{p}} \pi_i^k) \right) = \mathcal{L}_{\bar{p}}^{G, k}, \quad (1)$$

then each local joint invariant of k points can be locally expressed as a function of local joint invariants of $k - 1$,

$$\mathcal{A}^{G,k} = \pi_1^{k*} \mathcal{A}^{G,k-1} \odot \dots \odot \pi_k^{k*} \mathcal{A}^{G,k-1}.$$

Proof. We have that

$$(\pi_i^{k*}(\Sigma^{G,k-1}))^\perp = \mathcal{L}^{G,k} \oplus \ker(d\pi_i^k),$$

thus, the condition (1) is equivalent to:

$$\{\pi_1^{k*}(\Sigma^{G,k-1}), \pi_2^{k*}(\Sigma^{G,k-1}), \dots, \pi_i^{k*}(\Sigma^{G,k-1})\} = \Sigma^{G,k}$$

and then by Lemmas 1 and 2 we finish. \square

After integration, condition (1) have the following geometrical meaning. Let $U \subset W_k$ an small enough subset of an orbit in M^k . Then, U can be recovered as the intersection of its projections onto M^{k-1} :

$$U = \bigcap_{i=1}^k (\pi_i^k)^{-1}(\pi_i^k(U)).$$

We will see that condition (1) is always satisfied for locally faithful actions. If fact, for k big enough we only need to take into account two factors of the intersection, as the following elementary lemma of linear algebra shows.

Lemma 5 *Let A, B, C be real vector subspaces of a real vector space E . If $B \cap C = \{0\}$ and $A \cap (B \oplus C) = \{0\}$ then $(A \oplus B) \cap (A \oplus C) = A$.*

Proof. We prove only the non trivial inclusion. Let v be in $(A \oplus B) \cap (A \oplus C)$, then there are decompositions $v = a_1 + b = a_2 + c$ with $a_1, a_2 \in A$, $b \in B$ and $c \in C$. Let us take $w = a_1 - a_2 = c - b$, it is clear $w \in A \cap (B \oplus C)$ and then $w = 0$, it yields $a_1 = a_2$ and hence $b = c = 0$ so that $v = a_1$ what proves $v \in A$. \square

Theorem 2 (Finiteness theorem) *Assume that the action of G in M is locally faithful, and let k_0 be its local rank. There is a k_1 such that any local k -joint invariant in M points with $k > k_1$ is functionally dependent of the liftings of local k_1 -joint invariants. This number k_1 satisfies,*

$$k_1 \leq k_0 + 2.$$

Proof. Let us prove that for each $k > k_0 + 2$ the local k -joint invariants are functionally dependent of local $(k - 1)$ -joint invariant. By Lemmas 3 and 4 we consider a dense open subset W_k in M^k such that the action of G in M^k is locally regular each on $\pi_1^{k-1}(\pi_1^k(W_k)) \subset M^{k-2}$. For each \bar{p} in W_k we have that the spaces $\mathcal{L}_{\bar{p}}^{G,k}$, $\ker(d_{\bar{p}}\pi_1^k)$ and $\ker(d_{\bar{p}}\pi_2^k)$ satisfy the hypothesis of Lemma 5. Thus, we can apply Proposition 1, finishing the proof. \square

3 Weil near-points bundles

In order to compute differential invariants, we will use the formalism of Weil bundles. This is an approach to differential geometry proposed by Andre Weil in 1953 [8], who introduced the notion of infinitesimally near points. This formalism is an alternative to the better known notion of Jet bundles of Ehresmann. It has been developed independently by Shurygin [12], Kolář, Michor, Slovák [5], and Muñoz, Rodríguez, and Muriel [7]. The reader interested in the proofs of the statements and the details of the theory presented in this section is encouraged to consult the former references.

3.1 Infinitesimally near points

The spectral representation theorem says that there is a canonical bijection from M to $\text{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{C}^\infty(M), \mathbb{R})$. To each point $p \in M$ it corresponds the valuation morphism:

$$p: \mathcal{C}^\infty(M) \rightarrow \mathbb{R}, \quad f \mapsto p(f) := f(p).$$

Definition 7 A Weil algebra A is a finite dimensional local \mathbb{R} -algebra with maximal ideal \mathfrak{m}_A whose quotient field A/\mathfrak{m}_A is the field \mathbb{R} of real numbers. The quotient morphism $\omega_A: A \rightarrow \mathbb{R}$ is called the valuation morphism of A .

In the applications we mainly use a specific kind of Weil algebras, the algebras of truncated Taylor series of order r in m variables,

$$\mathbb{R}[[\varepsilon]]_{m,r} = \mathbb{R}[[\varepsilon]]/(\varepsilon)^{r+1}, \quad \varepsilon = (\varepsilon_1, \dots, \varepsilon_m).$$

Let us consider a Weil algebra A . We define the bundle of A -near-points² in M as $M(A) = \text{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{C}^\infty(M), A)$. By composition with the valuation ω_A we have a canonical projection $\pi_A: M(A) \rightarrow M$, which is a smooth bundle. For each open coordinate subset $U \subset M$ with coordinate functions,

$$\bar{x}: U \rightarrow W \subset \mathbb{R}^n$$

we have that $\pi_A^{-1}(U) = U(A)$ is endowed with coordinates with values in A ,

$$\bar{x}^A: U(A) \rightarrow A^n,$$

and that system of coordinates identify $U(A)$ with the open subset of A^n consisting in n -tuples $\bar{a} = (a_1, \dots, a_n)$ such that $(\omega(a_1), \dots, \omega(a_n))$ is in W .

We also consider the open sub-bundle $M(A)_{\text{prop}}$ consisting in the A -near points that are surjective as algebra morphisms. We have some interesting and self-explanatory examples:

- (a) The bundle $M(\mathbb{R}[[\varepsilon]]_{1,1}) \rightarrow M$ is the tangent bundle $TM \rightarrow M$. A vector $\vec{X}_p \in T_p X$ is seen as an \mathbb{R} -algebra morphism in the following way:

$$f \mapsto f(p) + \varepsilon \vec{X}_p f.$$

² $\mathbb{R}[[\varepsilon]]_{m,r}$ -near-points are simply termed (m, r) -near-points.

- (b) The bundle $M(\mathbb{R}[[\varepsilon]]_{m,1}) \rightarrow M$ is the bundle of frames $F(M) \rightarrow M$. A frame $(\vec{X}_{1p}, \dots, \vec{X}_{np})$ is seen as an \mathbb{R} -algebra morphism if the following way:

$$f \mapsto f(p) + \varepsilon_1 \vec{X}_{1p} f + \dots \varepsilon_n \vec{X}_{np} f.$$

- (c) The bundle $M(\mathbb{R}[[\varepsilon]]_{1,r}) \rightarrow M$ is the tangent bundle of order r , that we denote by $T^r M \rightarrow M$. An element $j_0^r \gamma$ of $T^r M$ is the Taylor development of order r at 0 of a curve $\gamma: (\mathbb{R}^m, 0) \rightarrow M$. If the expression in coordinates of γ is:

$$\gamma(\varepsilon) = (x_1(\varepsilon), \dots, x_n(\varepsilon)),$$

where,

$$\begin{cases} x_1(\varepsilon) = \gamma_1 + \gamma'_1 \varepsilon + \dots + \frac{\gamma_1^{(r)}}{r!} \varepsilon^r + o(\varepsilon^{r+1}), \\ \vdots \\ x_n(\varepsilon) = \gamma_n + \gamma'_n \varepsilon + \dots + \frac{\gamma_n^{(r)}}{r!} \varepsilon^r + o(\varepsilon^{r+1}), \end{cases}$$

then, $j_0^r \gamma$ is the morphism:

$$j_0^r \gamma: x_i \mapsto \gamma_i + \gamma'_i \varepsilon + \dots + \frac{\gamma_i^{(r)}}{r!} \varepsilon^r \in \mathbb{R}[[\varepsilon]]_{1,r}.$$

By convention we say that the coefficient $\gamma_i^{(j)}$ of the near-point is its $x_i^{(j)}$ coordinate. Thus, if x_1, \dots, x_n is a system of coordinates in $U \subseteq M$ then $x_1, \dots, x_n, x'_1, \dots, x'_n, \dots, x_1^{(r)}, \dots, x_n^{(r)}$ is a system of coordinates in $T^r U \subseteq T^r M$.

- (d) The bundle $M(\mathbb{R}[[\varepsilon]]_{m,r}) \rightarrow M$ is the generalized tangent bundle of order rank m and order r , that we denote by $T^{m,r} M \rightarrow M$. An element $j_0^r \varphi$ of $T^{m,r} M$ is the Taylor development of order r at 0 of a smooth map $\varphi: (\mathbb{R}^m, 0) \rightarrow M$. If the expression in coordinates of γ is:

$$\varphi(\varepsilon_1, \dots, \varepsilon_m) = (x_1(\varepsilon_1, \dots, \varepsilon_m), \dots, x_n(\varepsilon_1, \dots, \varepsilon_m)),$$

where,

$$\begin{cases} x_1(\varepsilon_1, \dots, \varepsilon_m) = \varphi_{1,0} + \sum_{1 \leq |\alpha| \leq r} \varphi_{1,\alpha} \frac{\varepsilon_1^{\alpha_1} \dots \varepsilon_m^{\alpha_m}}{\alpha_1! \dots \alpha_m!} + o(\varepsilon^{r+1}), \\ \vdots \\ x_n(\varepsilon_1, \dots, \varepsilon_m) = \varphi_{n,0} + \sum_{1 \leq |\alpha| \leq r} \varphi_{n,\alpha} \frac{\varepsilon_1^{\alpha_1} \dots \varepsilon_m^{\alpha_m}}{\alpha_1! \dots \alpha_m!} + o(\varepsilon^{r+1}), \end{cases}$$

then, $j_0^r \varphi$ is the morphism:

$$j_0^r \varphi: x_i \mapsto \varphi_{i,0} + \sum_{1 \leq |\alpha| \leq r} \varphi_{i,\alpha} \frac{\varepsilon_1^{\alpha_1} \dots \varepsilon_m^{\alpha_m}}{\alpha_1! \dots \alpha_m!} \in \mathbb{R}[[\varepsilon]]_{1,r}.$$

By convention we say that the coefficient $\varphi_{i,alpha}$ of the near-point is its $x_{i,\alpha}$ coordinate. Thus, if x_1, \dots, x_n is a system of coordinates in $U \subseteq M$ then x_1, \dots, x_n and $x_{i,\alpha}$ with $i = 1, \dots, n$, and $\alpha = (\alpha_1, \dots, \alpha_m)$ with $1 \leq |\alpha| \leq r$ form a a system of coordinates in $T^{m,r} U \subseteq T^{m,r} M$.

3.2 Multi-type near points

Definition 8 A *multi-Weil algebra* A is a finite direct product of Weil algebras. We will say that a point of

$$M(A) = \text{Hom}_{\mathbb{R}\text{-alg}}(\mathcal{C}^\infty(M), A)$$

is a multi-near-point of type A .

Multi-Weil algebras are not usually considered in the theory of near-points. However they are useful in order to relate joint and differential invariants. Given a multi-Weil algebra,

$$A = A_1 \times \dots \times A_k$$

the number k of factors is called the *multiplicity* of A . Note that $\text{Spec}(A)$ consist in k -maximal ideals $\{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$ and there are k valuation morphisms $\omega_{A_i}: A \rightarrow \mathbb{R}$. We denote by ω_A the map $(\omega_{A_1}, \dots, \omega_{A_k})$:

$$\omega_A: A \rightarrow \mathbb{R} \times \dots \times \mathbb{R}, \quad a \mapsto (\omega_{A_1}(a), \dots, \omega_{A_k}(a)).$$

Taking into account the the direct product is a categorical direct product for real algebras, we have that,

$$M(A_1 \times \dots \times A_k) = M(A_1) \times \dots \times M(A_k),$$

therefore multi-near-points are tuples of near-points. In particular we have

$$M(\mathbb{R} \times \dots \times \mathbb{R}) = M \times \dots \times M.$$

In this way we can consider tuples of points as an special case of multi-near-points. The composition with ω_A gives the natural projection $\pi_A: M(A) \rightarrow M^k$.

Proposition 2 Let $A = A_1 \times \dots \times A_k$ and $B = B_1 \times \dots \times B_s$ be multi-Weil algebras of multiplicity k and s respectively and let $\varphi: A \rightarrow B$ be a \mathbb{R} -algebra morphism. The composition with φ induces an smooth map $\varphi_*: M(A) \rightarrow M(B)$. This is a morphism of bundles in the following sense, there is a map

$$\sigma: \{1, \dots, s\} \rightarrow \{1, \dots, k\}$$

such that for any $p^A \in M(A)$,

$$\pi_B(p^A) = (\pi_{A\sigma(1)}(p^A), \dots, \pi_{A\sigma(s)}(p^A)).$$

Proof. It is easy to check that φ induces a morphism $\sigma^*: \mathbb{R}^k \rightarrow \mathbb{R}^s$ such that $\sigma^*(\omega_A(a)) = \omega_B(\varphi(a))$. Then, taking into account that $\text{Hom}_{\mathbb{R}\text{-alg}}(\mathbb{R}^k, \mathbb{R}^s)$ is the set of maps from $\{1, \dots, s\}$ to $\{1, \dots, k\}$ we finish. \square

3.3 Prolongation of functions

Let A be a multi-Weil algebra and $M(A)$ the bundle of A -multi-near-point in M . For each function $f \in \mathcal{C}_M^\infty(U)$ there is a prolongation of f to a A -valued function f^A in $M(A)$,

$$f^A: U(A) \rightarrow A, \quad p^A \mapsto f^A(p(A)) = p^A(f).$$

If we consider $\{a_1, \dots, a_s\}$ a basis of A , then, we can decompose f^A in real components:

$$f^A = \sum_{i=1}^s f_i a_i,$$

where each f_i is an smooth function in $A(U)$. The functions f_1, \dots, f_s are called the real components of f in $U(A)$ relative to the basis $\{a_1, \dots, a_s\}$.

Example 5 Let us consider $A = \mathbb{R}^k$ so that $M(\mathbb{R}^n) = M^k$. Let us consider $f \in \mathcal{C}^\infty(M)$. We have:

$$f^A(p_1, \dots, p_k) = (f(p_1), f(p_2), \dots, f(p_k)),$$

and the real components in the canonical basis of \mathbb{R}^k are,

$$f_i(p_1, \dots, p_k) = f(p_i).$$

Example 6 Let us consider $f \in \mathcal{C}^\infty(M)$, $A = \mathbb{R}[[\varepsilon]]_{1,r}$, and $\gamma: (\mathbb{R}, 0) \rightarrow M$ so that $M(A) = T^r M$ and $j_0^r(\gamma) \in T^r M$. We have:

$$f^A(j_0^r \gamma) = f(p) + \varepsilon \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} f(\gamma(\varepsilon)) + \dots + \frac{\varepsilon^r}{r!} \left. \frac{d^r}{d\varepsilon^r} \right|_{\varepsilon=0} f(\gamma(\varepsilon))$$

And the real components in the basis $\{1, \varepsilon, \dots, \varepsilon^r/r!\}$ are:

$$f^{(i)}(j_p^r \gamma) = \left. \frac{d^i}{d\varepsilon^i} \right|_{\varepsilon=0} f(\gamma(\varepsilon)).$$

In particular, for a system of coordinates x_1, \dots, x_n in M , their real components are the coordinate functions $x_i^{(j)}$ in $T^r M$.

3.4 Differential invariants

Let A be a (multi-)Weil algebra and $\phi: M \rightarrow M$ a diffeomorphism. Let us denote by ϕ^* the induced automorphism of $\mathcal{C}^\infty(M)$ defined as $\phi^*(f)(p) = f(\phi(p))$. The composition with ϕ^* naturally gives us an diffeomorphism of $M(A)$. This assignation compatible with the composition, and thus the action of G in M naturally prolongs to $M(A)$.

Example 7 For $A = \mathbb{R}^k$ we have $M(A) = M^k$ and this prolongation is simply the diagonal action of G in M^k .

Example 8 Let x_1, \dots, x_n be a system of coordinates in an open subset of M , and let (g_1, \dots, g_r) be a system of coordinates in a neighbourhood of the identity in G , the the action of G in M has an expression in coordinates:

$$\tilde{x}_1 = X_1(x, g), \dots, \tilde{x}_n = X_n(x, g).$$

The corresponding open subset of tangent space $T^r M$ of order r is coordinated by $x_1, \dots, x_n, x'_1, \dots, x'_n, \dots, x_1^{(r)}, \dots, x_n^{(r)}$. The action of G in $T^r M$ has an expression in coordinates that is computed iteratively:

$$\begin{aligned} \tilde{x}_j &= X_j(x, g), \\ \tilde{x}'_j &= X'_j(x, x', g) = \sum_{i=1}^n \frac{\partial X_j}{\partial x_i}(x, g) x'_i, \\ &\vdots \\ \tilde{x}_j^{(r)} &= X_j^{(r)}(x, x', \dots, x^{(r-1)}, g) = \sum_{i=1}^n \frac{\partial X_j^{(r-1)}}{\partial x_i}(x, g, x', \dots, x^{(r-1)}) x'_i. \end{aligned}$$

Given a Weil algebra A , a (local) differential invariant of type A is a (local) invariant of the action of G in $M(A)$. In particular, a local invariant of rank m and order of the action of G in M is a (local) invariant of the action of G in $T^{(m,r)} M$. There are formal differential operators called the *total derivatives*:

$$\frac{\mathfrak{d}}{\mathfrak{d}\varepsilon_j} : \mathcal{C}^\infty(T^{m,r} M) \rightarrow \mathcal{C}^\infty(T^{m,r+1} M)$$

whose expression in coordinates is:

$$\frac{\mathfrak{d}}{\mathfrak{d}\varepsilon_j} = \sum_{i=1}^n x_{i,e_j} \frac{\partial}{\partial x_i} + \sum_{|\alpha|>1} \sum_{i=1}^n x_{i,\alpha+e_j} \frac{\partial}{\partial x_{i,\alpha}}$$

which, for the case $m = 1$ has a simpler expression:

$$\frac{\mathfrak{d}}{\mathfrak{d}\varepsilon} = x'_1 \frac{\partial}{\partial x_1} + \dots + x'_n \frac{\partial}{\partial x_n} + x''_1 \frac{\partial}{\partial x'_1} + \dots + x''_n \frac{\partial}{\partial x'_n} + \dots$$

If I is a (local) differential invariant of rank m and order r , then its m total derivatives are (local) differential invariants of rank m and order $r + 1$. The classical finiteness theorem of Lie (given originally in [6, p. 760]) states:

Theorem 3 (Lie) *For each m there is an r_0 such that any local differential invariant of rank m and r with $r > r_0$ of the action of G in M is functionally dependent of differential invariants of rank m and order r_0 and their total derivatives.*

4 Derivation of joint invariants

4.1 Twisted differential

The following result is a simple, but important, observation. It is originally stated for in [8] for Weil algebras, but the proof remains the same for multi-Weil algebras.

Proposition 3 (Weil) *Let M be an smooth manifold, A, B multi-Weil algebras. There are canonical natural diffeomorphisms,*

$$M(A)(B) \simeq M(A \otimes_{\mathbb{R}} B) \otimes M(B)(A)$$

Proof. Let us consider $\{a_i\}_{i \in I}$ a basis of A , $\{b_j\}_{j \in J}$ basis of B . Then $\{a_i \times b_j\}_{I \times J}$ is a basis of $A \otimes_{\mathbb{R}} B$. Let $x_1 \dots, x_n$ be a system of coordinates in $U \subseteq M$. Then, the real components $x_{m,i}$, defined by:

$$x_m^A = \sum_{i \in I} x_{m,i} a_i,$$

form a system of coordinates in $U(A)$ and the real componets $x_{m,i,j}$ defined by:

$$x_{m,i}^A = \sum_{j \in J} x_{m,i,j} b_j$$

form a system of coordinates in $U(A)(B)$. On the other hand, the real components $x_{m,(i,j)}$ defined by:

$$x_m^{A \otimes_{\mathbb{R}} B} = \sum_{(i,j) \in I \times J} x_{m,(i,j)} a_i \otimes b_j$$

form a system of coordinates in $U(A \otimes_{\mathbb{R}} B)$. The desired diffeomorphism comes from the identification of the coordinates $x_{m,i,j}$ with $x_{m,(i,j)}$. \square

Corollary 1 *There are canonical natural diffeomorphisms:*

$$M(A)^k \simeq M(A^k) \simeq M^k(A).$$

Proof. Take $B = \mathbb{R}^k$ in Proposition 3. \square

Let $\bar{\sigma}$ be an r -algebra morphism $\bar{\sigma}: A \rightarrow A^k$. By the universal property of direct product, $\bar{\sigma}$ is a k -tuple of $\bar{\sigma} = (\sigma_1, \dots, \sigma_k)$ of morphisms from A to A ,

$$\bar{\sigma}(a) = (\sigma_1(a), \dots, \sigma_k(a)).$$

By Proposition 2 and Corollary 1, $\bar{\sigma}$ induces an smooth map

$$\bar{\sigma}_*: M(A) \rightarrow M(A^k) \simeq M^k(A),$$

also, for a function I defined in an open subset $U \subset M^k$ the A -prolongation of I is an A -valued function defined in $U(A) \subseteq M^k(A)$.

Definition 9 Let I be a function defined in some open subset U of M^k . We call the $\bar{\sigma}$ -twisted differential $D_{\bar{\sigma}}$ of I to the A -valued function $D_{\bar{\sigma}}I = I^A \circ \bar{\sigma}_*$.

Let us denote Δ^k the diagonal submanifold in M^k , and $\mathbf{di}_k: M \rightarrow M^k$ the diagonal map. Note that the domain of definition of $D_{\bar{\sigma}}I$ is $\mathbf{di}_k^{-1}U(A)$. Thus, the σ -twisted differential makes sense only for functions whose domain of definition intersects Δ_k .

Lemma 6 Let $U \subset M$ be an open subset and $I \in \mathcal{C}^\infty U$ be a (local) invariant of the action of G in M . For each multi-Weil algebra A , the A -valued function $I^A \in \mathcal{C}^\infty(U(A), A)$ is an A -valued (local) invariant the action of G in $M(A)$.

Proof. The assignation $M \rightsquigarrow M(A)$ is a functor. For each smooth map $\alpha_g: M \rightarrow M$ we have $(I \circ \alpha_g)^A = I \circ \alpha_g^A$. Hence, if I is an invariant then I^A is an invariant. \square

Lemma 7 Let $\sigma: A \rightarrow B$ be a morphism of multi-Weil algebras. Let $\sigma_*: M(A) \rightarrow M(B)$ be the induced morphism between the spaces of multi-near-points. Let E be a vector space, and $I \in \mathcal{C}^\infty(W, E)$ defined in $W \subset M(B)$ be a (local) invariant of the prolonged action of G in $M(B)$. Then $I \circ \sigma_*$ is a (local) invariant of the action of G in $M(A)$.

Proof. As the above lemma, since the assignation $A \rightsquigarrow M(A)$ is a functor, σ_* is a G -equivariant map between $M(A)$ and $M(B)$. \square

Theorem 4 If I is a (local) k -joint invariant of the action of G in M , then $D_{\bar{\sigma}}I$ is an A -valued (local) differential invariant of type A of the action of G in M .

Proof. First, we apply the Lemma 6 to the case of M^k , and then Lemma 7 to the case of $\bar{\sigma}$. \square

Thus, for any basis of A , the real components of $D_{\bar{\sigma}}I$ are local differential invariants of the action of G in M . In particular, for the case $A = \mathbb{R}[[\varepsilon]]_{1,r}$ the coefficient of the twisted derivative in $\varepsilon^r/r!$ is the r -th $\bar{\sigma}$ -twisted derivative of the invariant $\left(\frac{\partial^r}{\partial \varepsilon^r I}\right)_{\bar{\sigma}}$.

$$D_{\bar{\sigma}}I = \sum_{i=1}^r \frac{\varepsilon^i}{i!} \left(\frac{\partial^i I}{\partial \varepsilon^i} \right)_{\bar{\sigma}}.$$

Example 9 For the action of $\text{Mov}(\mathbb{R}^n)$ the function

$$d^2(p_1, p_2) = \sum_{i=1}^n (x_{i,1} - x_{i,2})^2$$

is a 2-joint invariant. Let us consider in $\mathbb{R}[[\varepsilon]]_{1,1}$ the family of endomorphisms,

$$\tau_i: \mathbb{R}[[\varepsilon]]_{1,1} \rightarrow \mathbb{R}[[\varepsilon]]_{1,1}, \quad \varepsilon \mapsto i\varepsilon.$$

And denote $\bar{\tau} = (\tau_0, \tau_1)$. The twisted derivative is computed in the following way. We set the $(1, 1)$ -near-point ,

$$p(\varepsilon) = (x_1 + \varepsilon x'_1, \dots, x_n + \varepsilon x'_n),$$

then,

$$D_{\bar{\tau}}(d^2)(p(\varepsilon)) = d(p(0), p(\varepsilon))^2,$$

Where this last expression is seen as a Taylor series of order one in ε . A direct computation yields:

$$D_{\bar{\tau}}(d^2) = \varepsilon \sum_{i=1}^n (x_i')^2,$$

and thus,

$$\left(\frac{\mathfrak{d}(d^2)}{\mathfrak{d}\varepsilon} \right)_{\bar{\tau}} = \sum_{i=1}^n (x_i')^2,$$

which is the infinitesimal quadratic expression of the euclidean metric, and it is a differential invariant of rank 1 and order 1.

Example 10 We consider the action $\text{Mov}(2, \mathbb{R})$ in \mathbb{R}^2 and the following invariant, proportional to the oriented area 3-joint invariant:

$$A(p_1, p_2, p_3) = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

And denote $\bar{\tau} = (\tau_0, \tau_1, \tau_2)$ defined as before, but as endomorphisms of $\mathbb{R}[[\varepsilon]]_{1,3}$. We set in $T^3\mathbb{R}^2$ the near-point:

$$p(\varepsilon) = \left(x + \varepsilon x' + \frac{\varepsilon^2}{2} x'' + \frac{\varepsilon^3}{6} x''', y + \varepsilon y' + \frac{\varepsilon^2}{2} y'' + \frac{\varepsilon^3}{6} y''' \right)$$

A direct computation of the twisted derivative yields:

$$D_{\bar{\tau}}A = A(p(0), p(\varepsilon), p(2\varepsilon)) = \varepsilon^3 \begin{vmatrix} x' & x'' \\ y' & y'' \end{vmatrix}.$$

And then,

$$\left(\frac{\mathfrak{d}A}{\mathfrak{d}\varepsilon} \right)_{\bar{\tau}} = 6(x'y'' - x''y'),$$

which is a differential invariant of rank 1 and order 2.

Example 11 The above example can be carried out in dimension n . We consider the action $\text{Mov}(n, \mathbb{R})$ in \mathbb{R}^n and the oriented volume $(n+1)$ -joint invariant:

$$V_n(p_1, \dots, p_{n-1}) = \det(p_2 - p_1, p_3 - p_1, \dots, p_{n+1} - p_1),$$

We consider $\bar{\tau} = (\tau_0, \tau_1, \dots, \tau_n)$ endomorphisms of $\mathbb{R}[[\varepsilon]]_{1,n+1}$ defined as in Example 9. We set in $T^{n+1}\mathbb{R}^n$ the near-point:

$$p(\varepsilon) = \left(x_1 + \varepsilon x'_1 + \dots + \frac{\varepsilon^{n+1}}{(n+1)!} x_1^{(n+1)}, \dots, x_n + \varepsilon x'_n + \dots + \frac{\varepsilon^{n+1}}{(n+1)!} x_n^{(n+1)} \right)$$

A direct computation of the twisted derivative yields:

$$D_{\bar{\tau}} V = V(p(0), p(\varepsilon), p(2\varepsilon), \dots, p(n\varepsilon)) = \varepsilon^{n+1} \Lambda_n \mathcal{W}(x'_1, \dots, x'_n)$$

where Λ_n is a constant that depends of n with value,

$$\Lambda_n = \frac{1}{1!2!\dots n!} \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 2 & 2^2 & 2^3 & \dots & 2^n \\ \vdots & & & & \vdots \\ n & n^2 & n^3 & \dots & n^n \end{vmatrix}.$$

And \mathcal{W} denotes the Wronskian determinant,

$$\mathcal{W}(x'_1, \dots, x'_n) = \begin{vmatrix} x'_1 & x''_1 & \dots & x_n^{(n)} \\ x'_2 & x''_2 & \dots & x_2^{(n)} \\ \vdots & & & \vdots \\ x'_n & x''_n & \dots & x_n^{(n)} \end{vmatrix},$$

and thus,

$$\left(\frac{\mathfrak{d}^{n+1} V_n}{\mathfrak{d} \varepsilon^{n+1}} \right)_{\bar{\tau}} = (n+1)! \Lambda_n \mathcal{W}(x'_1, \dots, x'_n)$$

which turns out to be a differential invariant of rank 1 and order n of the action of the movements.

4.2 Joint invariants that degenerate on the diagonal

Let us consider I, J two functions defined in some open subset U of M , then for each multi-Weil algebra A it is apparent than their prolongations to $U(A)$ in $\mathcal{C}^\infty(U(A), A)$ satisfy:

$$(IJ)^A = I^A J^A.$$

Here, in the left side of the equation we have the usual product of real functions, and in the right we have the pointwise product of elements of A .

Let us assume now that A is a Weil algebra. If J is a non-vanishing function, we also have that for each $p^A \in U(A)$ $J^A(p^A) \notin \mathfrak{m}_A$ and thus it is an invertible element. We have then that:

$$\left(\frac{I}{J} \right)^A = \frac{J^A}{I^A}.$$

However, in some cases the above expression makes sense even if J is vanishing. Let us consider a, b two elements of A such that $a \in (b)$, then, there set element

$c \in A$ such that $bc = a$ form a class in the quotient algebra $A/\mathbf{Ann}(b)$. Thus, we may interpret the fraction as:

$$\frac{a}{b} = c \in A/\mathbf{Ann}(b).$$

Applying the above argument to the case of a (local) k -joint invariant which is expressed as a fraction I/J where the denominator J vanishes along the diagonal Δ_k , we have the following result.

Theorem 5 *Let $W \subset U$ be open subsets in M^k such that:*

- (a) *W is dense in U .*
- (b) *U has non empty intersection with the diagonal Δ_k .*

Let us denote by $U' = \mathbf{di}_k^{-1}(U)$. Let I, J be smooth functions defined in U such that their quotient $\frac{I}{J}$ is well defined in W and it is a (local) k -joint invariant. Assume that there is an ideal \mathfrak{p} in A such that for each $p^A \in U'(A)$:

- (c) *$D_{\bar{\sigma}}I(p^A)$ is in the ideal $(D_{\bar{\sigma}}J(p^A))$.*
- (d) *$\mathbf{Ann}(D_{\bar{\sigma}}J(p^A)) \subseteq \mathfrak{p}$.*

Then, $\frac{D_{\bar{\sigma}}I}{D_{\bar{\sigma}}J}$ is a well defined A/\mathfrak{p} -valued function in $U(A)$, and it is a (local) A -differential invariant of the action of G in M .

Example 12 Let us consider the action of $\text{Aff}(1, \mathbb{R})$ in \mathbb{RP}_1 . and affine ration:

$$R(x_1, x_2, x_3) = \frac{x_3 - x_1}{x_2 - x_1} = \frac{I(x_1, x_3)}{J(x_1, x_2)}$$

We again consider $\bar{\tau} = (\tau_0, \tau_1, \tau_2)$ the previously defined 3-tuple of endomorphisms of $\mathbb{R}[[\varepsilon]]_{1,2}$. Let us consider $x(\varepsilon)$ a near point in $T^2\mathbb{RP}_1$, that is expressed in the affine coordinate and its derivatives up to second order:

$$x(\varepsilon) = x + \varepsilon x' + \frac{\varepsilon^2}{2} x''.$$

A direct computation of the twisted derivative yields:

$$D_{\bar{\tau}}I(x(\varepsilon)) = \varepsilon \left(x' + \varepsilon x'' + \frac{8\varepsilon^2}{6} x''' \right), \quad D_{\bar{\tau}}J(x(\varepsilon)) = \varepsilon \left(x' + \frac{\varepsilon}{2} x'' + \frac{\varepsilon^2}{6} x''' \right).$$

We are under the hypothesis of Theorem 5, and the quotient is defined in $\mathbb{R}[[\varepsilon]]_{1,2}/\mathbf{Ann}(\varepsilon) = \mathbb{R}[[\varepsilon]]_{1,1}$. In such Weil algebra we have:

$$\frac{D_{\bar{\tau}}I}{D_{\bar{\tau}}J} = \frac{x' + \varepsilon x''}{x' + \frac{\varepsilon}{2} x''} = (x' + \varepsilon x'') \left(\frac{1}{x'} - \frac{\varepsilon x''}{2x'^2} \right) = 1 + \frac{x''}{x'} \frac{\varepsilon}{2}.$$

It turns out that,

$$2 \left(\frac{\mathfrak{d}^2 R}{\mathfrak{d} \varepsilon^2} \right)_{\bar{\tau}} = \frac{x''}{x'},$$

is the logarithmic derivative, a differential invariant of rank 1 and order 2.

Example 13 Let us consider the action of $\mathrm{PGL}(2, \mathbb{R})$ in \mathbb{RP}_1 . and anharmonic ratio:

$$R(x_1, x_2, x_3) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_2)(x_3 - x_4)} = \frac{I(x_1, x_3, x_3, x_4)}{J(x_1, x_2, x_3, x_4)}$$

We take again $\bar{\tau} = (\tau_0, \tau_1, \tau_2, \tau_4)$ the previously defined 4-tuple of endomorphisms of $\mathbb{R}[[\varepsilon]]_{1,4}$. Let us consider $x(\varepsilon)$ a near point in $T^4\mathbb{RP}_1$, that is expressed in the affine coordinate and its derivatives up to second order:

$$x(\varepsilon) = x + \varepsilon x' + \frac{\varepsilon^2}{2}x'' + \frac{\varepsilon^3}{6}x''' + \frac{\varepsilon^4}{24}x^{(4)}.$$

A direct computation of the twisted derivative yields:

$$\begin{aligned} D_{\bar{\tau}}I(x(\varepsilon)) &= I(x(0), x(\varepsilon), x(2\varepsilon), x(3\varepsilon)) = \\ &= \varepsilon^2 \left(4x'^2 + 12x'x''\varepsilon + \frac{24x''^2 + 34x'x'''}{3}\varepsilon^2 + o_1\varepsilon^3 \right) \\ D_{\bar{\tau}}J(x(\varepsilon)) &= J(x(0), x(\varepsilon), x(2\varepsilon), x(3\varepsilon)) = \\ &= \varepsilon^2 \left(x'^2 + 3x'x''\varepsilon + \frac{15x''^2 + 40x'x'''}{12}\varepsilon^2 + o_2\varepsilon^3 \right) \end{aligned}$$

Were o_1 and o_2 are higher order terms that do not have effect in our further computations. We are again under the hypothesis of Theorem 5, and the quotient is defined in $\mathbb{R}[[\varepsilon]]_{1,4}/\mathbf{Ann}(\varepsilon^2) = \mathbb{R}[[\varepsilon]]_{1,2}$. In such Weil algebra we have:

$$\begin{aligned} \frac{D_{\bar{\tau}}I}{D_{\bar{\tau}}J} &= \frac{4x'^2 + 12x'x''\varepsilon + \frac{24x''^2 + 34x'x'''}{3}\varepsilon^2}{x'^2 + 3x'x''\varepsilon + \frac{15x''^2 + 40x'x'''}{12}\varepsilon^2} = \\ &= 4 + \frac{3x''^2 - 2x'''x'}{x'^2}\varepsilon^2. \end{aligned}$$

It turns out that,

$$\frac{1}{2} \left(\frac{\mathfrak{d}^2 R}{\mathfrak{d}\varepsilon^2} \right)_{\bar{\tau}} = \frac{3x''^2 - 2x'''x'}{x'^2},$$

is the Schwartzian derivative, a differential invariant of rank 1 and order 3.

Final Remarks

In this work we have developed part of the theory of joint invariants and explored the link between joint and differential invariants. There is several interesting open questions. One is the theory of rational joint invariants. It is true that for an algebraic actions the joint invariants of many points are rational functions of the liftings of a system of joint invariants? Another interesting question concerns to the actions of pseudogrops. Is there also a finiteness theorem? And finally to differential invariants. It is true that we can obtain a generating system of differential invariants by twisted derivatives of joint invariants?

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